Lecture 18

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1 Flow

1.0.1 Reminder

We are looking for a flow in a network with the maximum flux.

1.0.2 Algorithm - first attempt

Input: A flow network $N = (V, E, c, s, t)$

1. **Initialisation:** $c' = c, \ f = 0$
2. We will define $E_{c'} = \{e \in E : c'(e) > 0\}$ and $G_{c;} = (V, E_{c'})$
3. While there exists a directed path $p = (x_0 = s, \ x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k = t)$ between s and t in $G_{c'}$:
   (a) Let $c_p = \min_{e \in p} \{c'(e)\}$
   (b) We will define
   $$f_p(e) = \begin{cases} 0, & \text{if } e \notin p \\ 0, & \text{if } e \in p \end{cases}$$
   (c) We update $f = f + f_p$
   (d) We update $c'(e) = c(e) - f(e)$ for every $e \in E$

This is a nice algorithm, but it doesn't work. You can find an example in the lesson.

1.0.3 Algorithm - solution

Input: Let there be $N = (V, E, c, s, t)$ a normal network flow. Its extension will be $N' = (V, c', s, t)$, where $c' : V \times V \to \mathbb{R}^+$ is the extended capacity function defined as follows:

$$c'(x, y) = \begin{cases} c(x, y), & \text{if } (x, y) \in E \\ 0, & \text{if } (x, y) \notin E \end{cases}$$

Let there be $N = (V, E, c, s, t)$ a regular network flow, and $f : E \to \mathbb{R}^+$ a regular correct flow that enables the requirement that **there is no flow between two antiparallel edges**. The extension f' of f will be the function $f' : V \times V \to \mathbb{R}$, defined as follows:

$$f'(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in E \wedge f(x, y) > 0 \\ -f(x, y), & \text{if } (y, x) \in E \wedge f(y, x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

We will note that f' is well defined thanks to this requirement.

Let there be $N' = (V, c', s, t)$ an extended network flow. The extended flow f' in this network is the function $f' : V \times V \to \mathbb{R}$ such that the following three requirements are met:

1. **Anti symmetry:** $\forall x, y \in V, \ f'(x, y) = -f'(y, x)$
2. **Capacity constraint:** $\forall x, y \in V, \ f'(x, y) \leq c'(x, y)$
3. **Conservation of mass:** $\forall x \in V, \ x \neq s, t, \ \sum_{y \in V} f'(x, y) = 0$

Theorem 1 (Lemma)**.** *Let f be a regular flow in a regular flow network N, then the extension of f, f' is a correct extended flow in the extended network flow N', which is the extension of N.*

1.0.4 Flux in an extended flow

Let there be $N' = (V, c', s, t)$ an extended network flow. Let f' be an extended flow in this network. We will define the flux of f':

$$|f'| = \sum_{x \in V} f'(s, x)$$

Theorem 2 (Lemma)**.** *Let f be a regular flow in a regular network flow N, and f' be its extension in the extended graph N'. Then $|f'| = |f|$*

1.0.5 Contraction of a network flow

Let there be $N' = (V, c', s, t)$ an extended network flow. We will define the set of edges $E \subseteq V \times V$ as follows:

$$E = \{(x, y) \in V \times V : c'(x, y) > 0\}$$

We will define the capacity function $c : E \to \mathbb{R}^+$ as follows: $c(x, y) = c'(x, y)$ for every $(x, y) \in E$.
We will define $N = (V, E, c, s, t)$ as the contraction of N'.

Theorem 3 (Lemma)**.** *Let N be a regular network flow, N' an extension of N, and $\overline{N}$ the contraction of N', then $N = \overline{N}$*

1.0.6 Contraction of flow

Let there be $N' = (V, c', s, t)$ an extended network flow, and $f' : V \times V \to \mathbb{R}$ be an extended flow in this network. Let $N = (V, E, c, s, t)$ be the contraction of N'. We will define the **contraction of** f' as the function $f : E \to \mathbb{R}^+$ defined as follows for every $(x, y) \in E$:

$$f(x, y) = \max\{f'(x, y), 0\}$$

Theorem 4 (Lemma)**.** *Let there be f' will be an extended flow in the extended network flow N', let N be the contraction of N', and f be the contraction of f'. So f is a regular flow in the regular graph N and $|f| = |f'|$*