Lecture 19

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1 Network flow 4 - Ford Fulkerson

1.0.1 Extensions of a network flow

Let there be $N = (V, E, c, s, t)$ a normal network flow. Its extension will be $N' = (V, c', s, t)$, where $c' : V \times V \to \mathbb{R}^+$ is the extended capacity function defined as follows:

$$c'(x, y) = \begin{cases} c(x, y), & \text{if } (x, y) \in E \\ 0, & \text{if } (x, y) \notin E \end{cases}$$

Let there be $N = (V, E, c, s, t)$ a regular network flow, and $f : E \to \mathbb{R}^+$ a regular correct flow that enables the requirement that **there is no flow between two antiparallel edges**. The extension f' of f will be the function $f' : V \times V \to \mathbb{R}$, defined as follows:

$$f'(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in E \wedge f(x, y) > 0 \\ -f(x, y), & \text{if } (y, x) \in E \wedge f(y, x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

We will note that f' is well defined thanks to this requirement.

Let there be $N' = (V, c', s, t)$ an extended network flow. The extended flow f' in this network is the function $f' : V \times V \to \mathbb{R}$ such that the following three requirements are met:

1. **Anti symmetry:** $\forall x, y \in V,\ f'(x, y) = -f'(y, x)$
2. **Capacity constraint:** $\forall x, y \in V,\ f'(x, y) \leq c'(x, y)$
3. **Conservation of mass:** $\forall x \in V,\ x \neq s, t,\ \sum_{y \in V} f'(x, y) = 0$

1.0.2 Contraction of a network flow

Let there be $N' = (V, c', s, t)$ an extended network flow. We will define the set of edges $E \subseteq V \times V$ as follows:

$$E = \{(x, y) \in V \times V : c'(x, y) > 0\}$$

We will define the capacity function $c : E \to \mathbb{R}^+$ as follows: $c(x, y) = c'(x, y)$ for every $(x, y) \in E$.
We will define $N = (V, E, c, s, t)$ as the contraction of N'.

Theorem 1 (Lemma)**.** *Let N be a regular network flow, N' an extension of N, and $\overline{N}$ the contraction of N', then $N = \overline{N}$*

1.0.3 Contraction of flow

Let there be $N' = (V, c', s, t)$ an extended network flow, and $f' : V \times V \to \mathbb{R}$ be an extended flow in this network. Let $N = (V, E, c, s, t)$ be the contraction of N'. We will define the **contraction of** f' as the function $f : E \to \mathbb{R}^+$ defined as follows for every $(x, y) \in E$:

$$f(x, y) = \max\{f'(x, y), 0\}$$

Theorem 2 (Lemma)**.** *Let there be f' will be an extended flow in the extended network flow N', let N be the contraction of N', and f be the contraction of f'. So f is a regular flow in the regular graph N and $|f| = |f'|$*

1.0.4 Residual capacity and residual graph

Let $N' = (V, c', s, t)$ an extended network flow, and $f' : V \times V \to \mathbb{R}$ the extended flow in this network.

Definition 1.1 (Residual capacity)**.** *The **residual capacity** is the function $c_{f'} : V \times V \to \mathbb{R}^+$ defined as follows:*

$$\forall x, y \in V \ \ c_{f;}\left(x,y\right) = c'\left(x,y\right) - f'\left(x,y\right)$$

Definition 1.2 (Residual edges collection)**.**

$$\mathbb{E}_{f'} = \left\{(x,y) : c_{f'}\left(x,y\right) > 0\right\}$$

Definition 1.3 (Residual graph)**.**

$$G_{f'} = \left(V, E_{f'}\right)$$

1.0.5 Extended path and remaining flow

Definition 1.4 (Extended path)**.** *This is the simplest directed path between s and t in the residual graph*

Definition 1.5 (Residual capacity)**.** *Let P be the extended path. The **residual capacity** of P is*

$$c_{f''}\left(P\right) = \min_{e \in P}\left\{(c)f'\left(e\right)\right\}$$

Definition 1.6 (Residual flow)**.** $\Delta_{f',P} : V \times V \to \mathbb{R}$ *defined as follows:*

$$\Delta_{f',P}\left(x,y\right) = \begin{cases} c_{f'(P)}, & \textit{if } (x,y) \in P \\ -c_{f'(P)}, & \textit{if } (y,x) \in P \\ 0, & \textit{otherwise} \end{cases}$$

1.0.6 Algorithm Ford-Fulkerson for optimal flow in a network

Input: a Normal network flow $N = (V, E, c, s, t)$
Output: An optimal flow f in N

1. **Preprocessing:** Extension: We will extend N to the extended network N'
2. **Initialisation:** $f' = 0$, $(\forall x, y \in V, \ f'\left(x,y\right) = 0)$
3. **Iteration:** We find the extended path P in the residual graph $G_{f'}$, and update $f' = f' + \Delta_{f',P}$
4. **Stop:** When there are no remaining extended paths paths in the residual graph $G_{f'}$, we stop
5. **Contraction:** We contract the extended flow f' to a normal flow f, and return f

1.0.7 Algorithm proof

Theorem 3. *We will assume that the input of the algorithm FF is a network with integer capacities. So the algorithm stops after at most*

$$\sum_{x \in V} c\left(s,x\right)$$

iterations, and returns a flow with whole values.

Theorem 4. *If the algorithm FF stops, it returns an optimal flow*

Theorem 5 (Conclusion)**.** *Under the assumption that the input to FF is a network with integers, the algorithm stops and returns an optimal flow with integer values*

Theorem 6 (Lemma 1)**.** *Let N' be an extended network flow, and let f' be an extended flow in this network. Let P be the extended path in the residual graph $F_{f'}$, then $g = f' + \Delta_{f',P}$ is the extended flow in the network N', and additionally $|g| = |f'| + c_{f'}(P)$*

Proof . We will show that g is a correct extended flow:

1. Antisymmetry: According to the assumption f' enables antisymmetry, and according to the definition $\Delta_{f',P}$ enables antisymmetry. We will take $x, y \in V$:

$$\begin{aligned} g\left(x,y\right) &= f'\left(x,y\right) + \Delta_{f',P}\left(x,y\right) \\ &= -\left(f'\left(y,x\right) - \Delta_{f',P}\left(y,x\right)\right) \\ &= g\left(y,x\right) \end{aligned}$$

□