Lecture 22

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1 Edmonds-Karp

We will run FF, and every time we need to choose an extended path, we will choose the path with the minimum length of edges. We may find this path in time $O\left(|E| + |V|\right)$ through DFS/BFS.

Theorem 1. *Given an input $N = (V, E, c, s, t)$, the algorithm of EK stops after at most $|E|\,|V|$ iterations, and therefore the overall runtime is $O\left(|V|\,|E|\left(|V| + |E|\right)\right)$*

1.1 Definitions

Definition 1.1 (f'_i)**.** *We will use f'_i to be the extended flow in the network after i iterations of the algorithm*

Definition 1.2 (δ_i)**.** *For the node $x \in V$ we will use $\delta_i\left(x\right)$ to be the distance (length in edges of the minimal directed path) between s and x in $G_{f'_i}$*

Definition 1.3. *(x, y) will be called a* ***critical edge*** *in some iteration of the algorithm if it is an edge with* ***minimal residual capacity*** *in the chosen extended path in this iteration.*

1.2 Lemmas

Theorem 2 (Lemma 1)**.**

$$\forall x \in V \;\; \delta_0\left(x\right) \leq \delta_1\left(x\right) \leq \dots$$

Proof . In the course book □

Theorem 3 (Lemma 2)**.** *We will assume that the edge (x, y) is a critical edge during the iterations $i < j$ of the algorithm. Then $\delta_j\left(x\right) \geq \delta_i\left(x\right) + 2$*

Proof . Let there be i, the iteration where (x, y) is a critical edge, which is to say

$$\begin{aligned} f'_i\left(x, y\right) &= f'_{i-1}\left(x, y\right) + \Delta_{f'_{i-1}, p_i}\left(x, y\right) \\ &= f'_{i-1}\left(x, y\right) + c'_{f'_{i-1}}\left(x, y\right) \\ &= f'_{i-1}\left(y, y\right) + c'\left(x, y\right) - f'_{i-1}\left(x, y\right) \\ &= c'\left(x, y\right) \end{aligned}$$

and therefore,

$$\begin{aligned} c'_{f'_i}\left(x, y\right) &= c'\left(x, y\right) - f'_i\left(x, y\right) \\ &= 0 \end{aligned}$$

and (x, y) won't be in G'_{f_i}.
Since (x, y) appears in $G'_{f_{j-1}}$, so there must be an iteration $i < k < j$ such that (y, x) is part of the extended path in this iteration.

$$\begin{aligned} \delta_j\left(x\right) &\overset{\text{Lemma 1}}{\geq} \delta_k\left(x\right) \\ \text{Since we chose the path and slightly more} &= \delta_k\left(y\right) + 1 \\ &\overset{\text{Lemma 1}}{\geq} \delta_i\left(y\right) + 1 \\ &= \delta_i\left(x\right) + 1 + 1 \\ \text{Since we chose the shortest path to } y &= \delta_i\left(x\right) + 2 \end{aligned}$$

□

Theorem 4. *Given an input $N = (V, E, c, s, t)$, the algorithm of EK stops after at most $|E|\,|V|$ iterations, and therefore the overall runtime is $O\left(|V|\,|E|\,(|V| + |E|)\right)$*

Proof. In how many iterations can (x, y) be a **critical edge**?

$$\frac{|V| - 1}{2}$$

The number of edges that can be critical is at most $2\,|E|$, and from here the number of iterations it at most

$$2\,|E|\,\frac{|V - 1|}{2} \leq |E|\,|V|$$

□