Tutorial 10

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1 Cuts

Definition 1.1 (Cut S, T)**.** *In a flow network, a cut is $S, T \subseteq V$ such that $S \cap T = \emptyset \wedge V = S \cup T$. We generally refer to $t \in T \wedge s \in S$.*

Theorem 1 (Lemma)**.** *For every legal cut and flow it is true that*

$$|f| \leq c(S, T)$$
$$c(S, T) = \sum_{(u,v) \in (S \times T) \cap E} c(u, v)$$

Proof . In class □

Theorem 2 (Conclusion)**.** *If we find a flow f, and cut S, T, such that $|f| = c(S, T)$, then f is maximal and (S, T) has minimum capacity*

Proof . Proof by contradiction with the lemma □

Theorem 3 (Max-flow min-cut)**.** *There exists a correct f and a correct cut S, T such that*

$$|f| = c(S, T)$$

Theorem 4. *Let there be an optimal flow f. Then for every minimum cut in the graph, the edges are saturated.*

1.1 The investors and actors problem

Input:

- A placement of the actors $a \in A$ of size n. For every actors there is a salary $s_i \in \mathbb{R}_{\geq 0}$.
- The group of investors $B = \{b_1, \ldots, b_n\}$. Every investor has a subset $A_i \subseteq A$ of players he wants, and if all the actors in A_i will be in a given film, then he will invest d_i

Output: Subsets $A' \subseteq A,\ B' \subseteq B$ such that:

- **Correctness:** For every $b_i \in B'$ it is true that $A_i \subseteq A'$
- **Maximisation:** $p(A', B')$ is maximised, where $p(A', B') = \sum_{b_i \in B'} d_i - \sum_{a_i \in A'} s_i$

1.1.1 Algorithm

We will build the network such that:

$$V = \{s, t\} \cup A \cup B$$
$$E = E_A \cup E_B \cup E_{AB}$$
$$E_{AB} = \{(b_i, a_j) | a_j \in A_i\}$$
$$E_B = \{(s, b_i) | b_i \in B\}$$
$$E_A = \{(a_j, t) | a_j \in A\}$$
$$c(u, v) = \begin{cases} d_i, & \text{if } (u, v) \in E_B,\ v = b_i \\ s_j, & \text{if } (u, v) \in E_A,\ u = a_j \\ \infty, & \text{if } (u, v) \in E_{AB} \end{cases}$$

Theorem 5. *The returned A', B' are correct.*

Proof . We will define a one to one, "on" matching between the cut (S, T), and the set (A', B'). Given S, T, we will define $B' = S \cap B$, $A' = S \cap A$. Given A', B' we can define $S = A' \cup B' \cup \{S\}$, and $T = V \setminus S$.

Theorem 6. *Given the matching, (S, T) are finite **if and only if** A', B' are correct.*

Proof . $c(S, T)$ is finite $\Leftrightarrow$ there does not exist $e \in E_{AB}$ (an edge with infinite capacity) that is cut
$\Leftrightarrow$ every edge in E_{AB} is either in T or in S
$\Leftrightarrow$ for every $b_i \in B$ it is true that A_i is located on the same side of the cut
$\Leftrightarrow$ we have not separated between b_i, A_i when they are located on one of the sides of the cut
$\Leftrightarrow$ $S \cap B,\ S \cap A$ defines the correct set. □

Theorem 7. *The minimum cut is finite*

Proof. There exists a finite cut (for example $S = \{s\}, T = V \setminus S$), with capacity $c(S, T) = \sum_{i=1}^{k} d_i$. The minimum cut is required to have capacity less than or equal to this (since it is minimal), and is therefore finite. □

Theorem 8 (Conclusion)**.** *According to the previous theorem, A', B' that match the minimum cut are correct*

□

Theorem 9. *The returned A', B' are optimal. $p(A', B') = \sum_{b_i \in B'} d_i - \sum_{a_j \in A'} s_j$ is maximal.*

1. *We will prove: $c(S, T) = D - p(A', B')$ where D is a constant*
2. *We will prove: $C(S, T)$ is minimal $\implies$ A', B' is maximised*

Proof. 1. Let there be (S, T), a finite cut:

$$
\begin{aligned}
c(S, T) &\overset{def}{=} \sum_{(u,v) \in (S \times T) \cap E} c(u, v) \\
&\overset{\text{Finite cut}}{=} \sum_{b_i \notin S} c(S, b_i) + \sum_{a_j \in S} c(a_j, t) \\
&= \sum_{b_i \in B} c(s, b_i) - \sum_{b_i \in B'} c(s, b_i) + \sum_{a_j \in A'} c(a_j, t) \\
&= \sum_{b_i \in B} d_i - \left(\sum_{b_i \in B'} d_i - \sum_{a_j \in A'} s_j \right) \\
&= D - p(A', B')
\end{aligned}
$$

2. If we assume by contradiction that $\widetilde{A}, \widetilde{B}$ are better than A', B'

$$
\begin{aligned}
p\left(\widetilde{A}, \widetilde{B}\right) &> p(A', B') \\
-p\left(\widetilde{A}, \widetilde{B}\right) &< -p(A', B') \\
D - p\left(\widetilde{A}, \widetilde{B}\right) &< D - p(A', B') \\
c\left(\widetilde{S}, \widetilde{T}\right) &< c(S, T)
\end{aligned}
$$

which is a contradiction of the minimality of the cut.

□

1.1.2 RunTime

1. Build the network:

$$\begin{aligned} |E| &= |E_B| + |E_A| + |E_{AB}| \\ &= k + n + nk \\ &= O(nk) \\ |V| &= n + k + 2 \\ &= O(n + k) \end{aligned}$$

2. Finding the cut: $O\left(|E|^2 |V|\right)$

3. Building the sets: $O(|V|)$

4. Total:

$$\begin{aligned} O\left(|E|^2 |V|\right) &= O\left(n^2 k^2 (n + k)\right) \\ &= O\left(n^3 k^2 + n^2 k^3\right) \end{aligned}$$

From here there was an example of EK (in the video), or FF (in their printed copy)